

# Plane Jacobian conjecture for simple polynomials

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## Abstract

A non-zero constant Jacobian polynomial map  $F = (P, Q) : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$  has a polynomial inverse if the component  $P$  is a simple polynomial, i.e. if, when  $P$  extended to a morphism  $p : X \longrightarrow \mathbb{P}^1$  of a compactification  $X$  of  $\mathbb{C}^2$ , the restriction of  $p$  to each irreducible component  $C$  of the compactification divisor  $D = X - \mathbb{C}^2$  is either degree 0 or 1.

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**1.** Let  $F = (P, Q) : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$  be a polynomial map,  $P, Q \in \mathbb{C}[x, y]$ , and denote  $JF := P_x Q_y - P_y Q_x$  the Jacobian of  $F$ . The mysterious Jacobian conjecture (JC) (See [4] and [2]), posed first by Keller in 1939 and still open, asserts that  $F$  has a polynomial inverse if the Jacobian  $JF$  is a non-zero constant. In 1979 by an algebraic approach Razas [17] proved this conjecture for the most simple geometrical case when  $P$  is a rational polynomial, i.e the generic fiber of  $P$  is a punctured sphere, and all fibres  $P = c$ ,  $c \in \mathbb{C}$ , are irreducible. In attempt to understand the geometrical nature of (JC), this case was also reproved by Heitmann [5] and Lê and Weber [10] in some other approaches. In fact, as observed by Neumann and Norbudy in [12], every rational polynomial with all irreducible fibres is equivalent to the coordinate

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polynomial. Most recent, Lê in [7] and [8] present the following observation, which was announced in the Hanoi conference, 2006, and the Kyoto conference, 2007.

**Theorem 1.** (Theorem 3.2 and Corollary 3.8 in [8]) *A non-zero constant Jacobian polynomial map  $F = (P, Q)$  has a polynomial inverse if  $P$  is a simple rational polynomial.*

Here, following [11], a polynomial map  $P : \mathbb{C}^2 \rightarrow \mathbb{C}$  is *simple* if, when extended  $P$  to a morphism  $p : X \rightarrow \mathbb{P}^1$  of a compactification  $X$  of  $\mathbb{C}^2$ , the restriction of  $p$  to each irreducible component  $\ell$  of the compactification divisor  $D = X - \mathbb{C}^2$  is either of degree 0 or 1. In fact, as in the proof in [8] of Theorem 1, if a component of non-zero constant Jacobian map  $F = (P, Q)$  is a simple rational polynomial, then this component determines a locally trivial fibration.

In this short paper we would like to present an other explanation for Theorem 1 from view point of the geometry of the non-proper valuer set of the map  $F$ . In fact, we shall prove

**Theorem 2.** *A non-zero constant Jacobian polynomial map  $F = (P, Q)$  has a polynomial inverse if  $P$  is a simple polynomial.*

In any meaning, the addition condition on the simple polynomial component in a non-zero constant Jacobian polynomial map may be viewed as a kind of “good” local conditions at infinity, but it seems to be not a global one. A completed classification of all simple rational polynomials was presented in [11].

**2.** Given a polynomial map  $F = (P, Q)$  of  $\mathbb{C}^2$ . Following [6], the non-proper value set  $A_F$  of  $F$  is the set of all values  $a \in \mathbb{C}^2$  such that there exists a sequence  $\mathbb{C}^2 \ni b_i \rightarrow \infty$  with  $F(b_i) \rightarrow a$ . This set  $A_F$  is either empty or an algebraic curve in  $\mathbb{C}^2$  for which every irreducible component is the image of a non-constant polynomial map from  $\mathbb{C}$  into  $\mathbb{C}^2$ . Our argument in the proof of Theorem 2 here is based on the following facts, that was presented in [13] and can be reduced from [3] (see also [14] and [15] for other refine versions).

**Theorem 3.** *Support  $F = (P, Q)$  is a polynomial map with non-zero constant Jacobian. If  $A_F \neq \emptyset$ , then every irreducible components of  $A_f$  can be parameterized by polynomial maps  $\xi \mapsto (\varphi(\xi), \psi(\xi))$  with*

$$\deg \varphi / \deg \psi = \deg P / \deg Q. \quad (1)$$

This theorem together with the Abhyankar-Moh Theorem [1] on the embeddings of the line to the plane allows us to obtain:

**Theorem 4.** *A polynomial map  $F$  of  $\mathbb{C}^2$  must have singularities if its non-proper value set  $A_F$  has an irreducible component isomorphic to the line.*

A simple proof of Theorem 4 recently presented in [15] gives a description in terms of Newton-Puiseux data how the singularity occurs in this situation.

**3.** To use Theorem 4 in the situation of simple polynomials, at first, we need to describe the non-proper value curve  $A_F$  in terms of the regular extension of  $F$  in a convenience compactification  $X \supset \mathbb{C}^2$ . Given a polynomial  $F = (P, Q)$ , extend it to a map  $F : \mathbb{P}^2 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$  and resolve the points of indeterminacy to get a regular map  $f = (p, q) : X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$  that coincides with  $F = (P, Q)$  on  $\mathbb{C}^2 \subset X$ . We call  $D = X - \mathbb{C}^2$  the divisor at infinity. The divisor  $D$  consists of a connected union of rational curves isomorphism to  $\mathbb{P}^1$  and the dual graph of the divisor  $D$  is a tree. An irreducible component  $\ell$  of  $D$  is a *horizontal* curve of  $P$  (or  $Q$ ) if the restriction of  $p$  (res.  $q$ ) to  $\ell$  is not a constant mapping. An irreducible component  $\ell$  of  $D$  is a *dicritical* curve of  $F$  if the restriction of  $f$  to  $\ell$  is not a constant mapping. A dicritical curve of  $F$  must be a horizontal curve of  $P$  or  $Q$ . Although the compactification defined above is not unique, the horizontal curves of  $P$  or  $Q$  as well as the dicritical curves of  $F$  are essentially independent of choice. Further, by blow-down components of self-intersection  $-1$  corresponding to linear vertexes or endpoint in the dual graph of  $D$ , but not of dicritical curves of  $F$ , horizontal curves of  $P$  or  $Q$ , we can work with minimal compactification  $X \supset \mathbb{C}^2$  on which  $F$  can be extended to a regular morphism  $f = (p, q) : X \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$ .

Let us denote  $D_\infty := f^{-1}((\{\infty\} \times \mathbb{P}^1) \cup (\mathbb{P}^1 \times \{\infty\}))$ . The following description of the dual graph of the divisor  $D$  is well-known (see, for example, in [18], [16] and [9]).

**Proposition 1.** *i) The dual graph of the divisor  $D$  is a tree;*

*ii) The dual graph of the curve  $D_\infty$  is a tree;*

*iii) The dual graph of each connected component of the closure of  $(D - D_\infty)$  is a linear path of the form*

$$\odot \rightarrow \circ \rightarrow \circ \rightarrow \cdots \circ \rightarrow \circ$$

*in which the beginning vertex  $\odot$  is a dicritical curve of  $F$  and the next possible vertexes  $\circ$  are curves to which the restriction of  $f$  are finite constant mappings.*

The following provides a description of the non-proper value set  $A_F$  of  $F$  in terms of regular extension of  $F$  in a minimal compactification  $X \supset \mathbb{C}^2$ .

**Proposition 2.** *i)*

$$A_F = \bigcup_{\text{dicritical curves } \ell \text{ of } F} f(\ell) \cap \mathbb{C}^2. \quad (2)$$

*ii)* Let  $\ell$  be a dicritical curve of  $F$ . Then,  $\ell$  and the curve  $D_\infty$  have a unique common point. Let  $\ell^* := \ell - D_\infty$ . Then, the curve  $\ell^*$  is isomorphic to  $\mathbb{C}$  and

$$f(\ell^*) = f(\ell) \cap \mathbb{C}^2 \quad (3)$$

*iii)*

$$A_F = \bigcup_{\text{dicritical curves } \ell \text{ of } F} f(\ell^*). \quad (4)$$

*Proof.* Conclusion (i) can be easily verified by the definition of  $A_F$  and a simple topological argument. Conclusion (ii) follows from the fact in Proposition 1 that the dual graph of the divisor  $D$  is a tree. Conclusion (iii) results from (i) and (ii).  $\square$

**4.** Now, we consider the situation when the restriction of  $p$  to a dicritical curve  $\ell$  of  $F$  is of degree 1.

**Lemma 1.** *Let  $\ell$  be a dicritical component of  $F$ . If the restriction of  $p$  to  $\ell$  is of degree 1, then the image  $f(\ell^*)$  is isomorphic to the line  $\mathbb{C}$ .*

*Proof.* Suppose  $\ell$  is a dicritical component of  $F$  and the degree of the restriction  $p|_\ell$  equals 1. Then,  $p|_\ell : \ell \rightarrow \mathbb{P}^1$  is injective, and hence, is bijective, since  $\ell$  is isomorphic to  $\mathbb{P}^1$ . This ensures that the curve  $f(\ell^*)$  intersects each line  $\{(u, v) \in \mathbb{C}^2 : u = c\}$ ,  $c \in \mathbb{C}$ , at a unique point. Then, the polynomial  $H(u, v)$  defining the curve  $f(\ell^*) \subset \mathbb{C}^2$  can be chosen of the form  $v + h(u)$ ,  $h \in \mathbb{C}[u]$ . So, the automorphism  $A(u, v) := (u, v - h(u))$  maps isomorphically the curve  $f(\ell^*)$  onto the line  $v = 0$ .  $\square$

*Proof of Theorem 2.* Suppose  $F = (P, Q)$  with  $JF \equiv c \neq 0$  and  $P$  is a simple polynomial. Note that each dicritical curve of  $F$  must be a horizontal curve of  $P$  or  $Q$ . Since  $JF \equiv c \neq 0$  and  $P$  is simple, in view of Theorem

4 and Lemma 1, a horizontal curve of  $P$  cannot be a dicritical curve of  $F$ . So, if  $\ell$  is a dicritical curve of  $F$ , then  $\ell$  must be a horizontal curve of  $Q$  and the restriction  $p|_{\ell}$  maps  $\ell$  to a finite constant. Thus, for such  $\ell$  the image  $f(\ell^*)$  is a line  $u = \text{const.}$ . The last is impossible again by Theorem 4 as  $JF \equiv c \neq 0$ . Hence,  $F$  has not any dicritical component. Then,  $A_F = \emptyset$  by Proposition 2 and  $F$  is a proper map by the definition of  $A_F$ . Therefore, by simple connectedness of  $\mathbb{C}^2$  the local diffeomorphic map  $F$  must be bijective. Thus,  $F$  is an automorphism of  $\mathbb{C}^2$ .

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